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ON THE EXTREMUM COMPLEMENTARY ENERGY PRINCIPLES FOR NONLINEAR ELASTIC SHELLS

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Abstract—The complementary variational problem is studied for thin elastic shells undergoing large deflections. By using the theory of convex analysis, a truly complementary extremum variational principle is derived as the dual problem. It is proved that both the minimum property of the complementary variational problem and the saddle point property of the generalized variational principle are closely related to a so-called dual gap function.

1. INTRODUCTION

The equations governing thin elastic shells undergoing moderate rotations about tangents and small rotations about normals to the midsurface are in general difficult to solve and it is desirable to develop approximate methods. The primal-dual variational principles provide powerful techniques for obtaining approximate solutions and they can also be used to generate numerical methods. As the primal problem, the stationary potential variational principles have received much attention recently, (c.f. e.g. Schmidt and Pietraszkiewicz, 1981; Iura, 1986; Szwabowicz, 1986). The stationary complementary principles in terms of the first Piola stress were studied by Stumpf (1979) and Wempner (1986). More recently, the primal extremum problem for geometrical nonlinear elastic thin shells has been studied by Wempner (1986) and the extremum potential variational principles are established by Gao (1989) based on convex analysis.

The dual extremum problems, however, are of great difficulty in large deformation cases. It is well known that the principle of complementary energy for infinitesimal deformation is usually derived from the potential energy principle by using the Legendre transformation (or Fenchel transformation if the total potential functional is nonsmooth). In other words, the total complementary energy is just the conjugate functional of the total potential energy. The symmetry between the primal and dual problems is amazingly beautiful. But, in geometric nonlinear cases, this symmetry was unfortunately broken. It has been proved by Gao and Strang (1988) that there exists a dual gap between the conjugate functional of the total potential energy and the total complementary potential energy, due to the nonlinearity of geometrical operator. It is found that the so-called dual gap function plays a key role in the analysis of geometrical nonlinear mechanics (Gao and Strang, 1989; Gao, 1989).

In the present work, a truly extremum complementary energy variational principle is established for geometrically nonlinear elastic shells. We will find that the extremum properties of the dual problem and the saddle point property of the generalized variational principle are closely related to this gap function.

2. PRELIMINARY RELATIONS

Let $\mathbf{r}(\xi^\alpha)$ be the position vector of the undeformed shell midsurface S, with curvilinear Gaussian coordinates $\xi^\alpha(\alpha = 1, 2)$ and covariant surface base vectors $\mathbf{a}_\alpha = \mathbf{r}_{,\alpha}$. The fields

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inherent in the surface S are $\mathbf{a} = \{a_{\alpha\beta}\} = \mathbf{a}_\alpha \otimes \mathbf{a}_\beta$—the metric tensor, $\mathbf{n} = \frac{1}{2}\varepsilon^{\alpha\beta}\mathbf{a}_\alpha \otimes \mathbf{a}_\beta$—the unit normal vector, $e = \varepsilon^{\alpha\beta}\mathbf{a}_\alpha \otimes \mathbf{a}_\beta$—the surface permutation tensor, $\mathbf{b} = \{b_{\alpha\beta}\} = \mathbf{a}_{\alpha,\beta}\cdot\mathbf{n}$—the curvature tensor. Using the Kronecker delta δ^α_β, the contravariant components of the metric tensor are defined by the relations $a^{\alpha\lambda}a_{\lambda\beta} = \delta^\alpha_\beta$.

Let us denote by a bar quantities referred to the deformed midsurface $\bar{S}$. The deformation of the shell midsurface from the undeformed reference configuration S into the deformed configuration $\bar{S}$ can be described by the displacement vector

$$\mathbf{u} = \bar{\mathbf{r}} - \mathbf{r} = u^\alpha \mathbf{a}_\alpha + w\mathbf{n}. \tag{1}$$

For the base vectors, we have the following relations Koiter (1966):

$$\bar{\mathbf{a}}_\alpha = l_{\lambda\alpha}\mathbf{a}^\lambda + \varphi_\alpha \mathbf{n} = \mathbf{a}_\alpha + \mathbf{u}_{,\alpha} \tag{2a}$$

$$\bar{\mathbf{n}} = n_\alpha \mathbf{a}^\alpha + n\mathbf{n} \tag{2b}$$

$$l_{\alpha\beta} = a_{\alpha\beta} + \vartheta_{\alpha\beta} - \omega_{\alpha\beta} \tag{2c}$$

$$\vartheta_{\alpha\beta} = \tfrac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) - b_{\alpha\beta}w \tag{2d}$$

$$\omega_{\alpha\beta} = \tfrac{1}{2}(u_{\beta|\alpha} - u_{\alpha|\beta}) \tag{2e}$$

$$\varphi_\alpha = w_{,\alpha} + b^\lambda_\alpha u_\lambda \tag{2f}$$

where $(\ )|_\alpha$ denotes the surface covariant differentiation at S

$$u_{\alpha|\beta} = u_{\alpha,\beta} - \Gamma^\lambda_{\alpha\beta}u_\lambda$$

$$w_{|\alpha\beta} = w_{,\alpha\beta} - \Gamma^\lambda_{\alpha\beta}w_{,\lambda}.$$

$\Gamma^\lambda_{\alpha\beta}$ is the Christoffel symbol of the undeformed shell midsurface S.

Assuming that the Kirchhoff–Love hypothesis holds, the midsurface strain tensor $\gamma = \{\gamma_{\alpha\beta}\}$ and the tensor of change of curvature $\kappa = \{\kappa_{\alpha\beta}\}$ can be expressed in the following:

$$\gamma_{\alpha\beta} = \tfrac{1}{2}(\bar{a}_{\alpha\beta} - a_{\alpha\beta}) \tag{3a}$$

$$\kappa_{\alpha\beta} = -(\bar{b}_{\alpha\beta} - b_{\alpha\beta}). \tag{3b}$$

Substituting eqns (2) into eqns (3) yields the strain–displacement relations. In the particular case of shells with moderately large rotations around tangents, we have:

$$\gamma_{\alpha\beta}(u_\alpha, w) = \vartheta_{\alpha\beta}(u_\alpha, w) + \tfrac{1}{2}\varphi_\alpha(u_\alpha, w)\varphi_\beta(u_\alpha, w) \tag{4a}$$

$$\begin{aligned}\kappa_{\alpha\beta}(u_\alpha, w) &= -\tfrac{1}{2}(\varphi_{\alpha|\beta} + \varphi_{\beta|\alpha}) - \tfrac{1}{2}b^\lambda_\alpha u_{\lambda|\beta} - \tfrac{1}{2}b^\lambda_\beta u_{\lambda|\alpha} + b^\lambda_\alpha b_{\lambda\beta}w \\ &= -w_{|\alpha\beta} - b^\lambda_{\alpha|\beta}u_\lambda - b^\lambda_\alpha u_{\lambda|\beta} - b^\lambda_\beta u_{\lambda|\alpha} + b^\lambda_\alpha b_{\lambda\beta}w.\end{aligned} \tag{4b}$$

We note that the strain tensor γ is nonlinear, but not the curvature tensor κ.

For elastic thin shells, the stored energy function $W(\gamma, \kappa)$ is quadratic in γ and κ:

$$W(\gamma, \kappa) = \tfrac{1}{2}H^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}\gamma_{\lambda\mu} + \tfrac{1}{2}h^{\alpha\beta\lambda\mu}\kappa_{\alpha\beta}\kappa_{\lambda\mu} \tag{5}$$

where

$$H^{\alpha\beta\lambda\mu} = H^{\beta\alpha\lambda\mu} = H^{\lambda\mu\alpha\beta}, \quad h^{\alpha\beta\lambda\mu} = h^{\beta\alpha\lambda\mu} = h^{\lambda\mu\alpha\beta}.$$

For isotropic shells we have

$$H^{\alpha\beta\lambda\mu} = \frac{Eh}{2(1+\nu)}\left(a^{\alpha\lambda}a^{\beta\mu}+a^{\alpha\mu}a^{\beta\lambda}+\frac{2\nu}{1-\nu}a^{\alpha\beta}a^{\lambda\mu}\right)$$

$$h^{\alpha\beta\lambda\mu} = \frac{h^2}{12}H^{\alpha\beta\lambda\mu}.$$

Here h is the shell thickness, E and ν denote the Young's modulus and Poisson's ratio, respectively. So the constitutive equations can be given as

$$N^{\alpha\beta} = \frac{\partial W}{\partial \gamma_{\alpha\beta}} = H^{\alpha\beta\lambda\mu}\gamma_{\lambda\mu} \tag{6a}$$

$$M^{\alpha\beta} = \frac{\partial W}{\partial \kappa_{\alpha\beta}} = h^{\alpha\beta\lambda\mu}\kappa_{\lambda\mu} \tag{6b}$$

in which $\mathbf{N} = \{N_{\alpha\beta}\}$, $\mathbf{M} = \{M_{\alpha\beta}\}$ denote the membrane force tensor and the bending moment tensor, respectively.

Let the vector $\bar{\mathbf{p}} = \bar{p}^\alpha \mathbf{a}_\alpha + \bar{p}\mathbf{n}$ be the loading distributed over the shell midsurface, $\bar{\mathbf{P}} = \bar{P}^\alpha \mathbf{a}_\alpha + \bar{P}\mathbf{n}$ the boundary force. On Γ_t, the bending moment $\bar{M}_n$ is also prescribed. Then the equilibrium equations can be given as follows:

$$(N^{\alpha\beta}-b^\alpha_\lambda M^{\lambda\beta})_{|\beta} - b^\alpha_\lambda(N^{\lambda\beta}\varphi_\beta + M^{\lambda\beta}_{|\beta}) + \bar{p}^\alpha = 0 \quad \text{in } S \tag{7a}$$

$$M^{\alpha\beta}_{|\alpha\beta} + b_{\alpha\beta}(N^{\alpha\beta}-b^\alpha_\lambda M^{\lambda\beta}) + (N^{\alpha\beta}\varphi_\beta)_{|\alpha} + \bar{p} = 0 \quad \text{in } S. \tag{7b}$$

Let $\Gamma = \partial S$ denote the boundary of S and $\Gamma = \Gamma_u \cup \Gamma_t$. For the sake of simplicity, it is assumed in this paper that along Γ_u the shell is clamped, i.e.

$$u_\alpha = 0, \quad w = 0, \quad \frac{\partial w}{\partial n} = 0 \quad \text{on } \Gamma_u. \tag{8}$$

On Γ_t the static boundary conditions are prescribed:

$$(N^{\alpha\beta}-b^\alpha_\lambda M^{\lambda\beta})n_\beta - b^\alpha_\beta M^{\lambda\beta}n_\lambda - \bar{P}^\alpha = 0 \quad \text{on } \Gamma_t \tag{9a}$$

$$(N^{\alpha\beta}\varphi_\beta + M^{\alpha\beta}_{|\beta})n_\alpha + \frac{\partial}{\partial s}(M^{\alpha\beta}t_\alpha n_\beta) - \bar{P} = 0 \quad \text{on } \Gamma_t \tag{9b}$$

$$M^{\alpha\beta}n_\alpha n_\beta - \bar{M}_n = 0 \quad \text{on } \Gamma_t \tag{9c}$$

where t, s, n are india of the boundary triad on Γ.

3. ABSTRACT DESCRIPTION AND PRIMAL PROBLEM

Even with the use of the tensorial notation, the governing equations for geometric nonlinear shells are rather complex and lengthy. For the purpose of convenience in description and deduction, the abstract notation from (Gao, 1989; Gao and Strang, 1988) will be used throughout this paper. Let $\mathscr{U}$ be an admissible displacements space, the elements in $\mathscr{U}$ are displacement $\mathbf{u} \in \mathscr{U}$. $\mathscr{E}$ the generalized strain space,

$$\mathbf{E} = \begin{Bmatrix}\gamma \\ \kappa\end{Bmatrix} \in \mathscr{E}$$

are generalized strain tensors. $\Lambda : \mathscr{U} \to \mathscr{E}$ is a geometrical nonlinear operator:

$$\Lambda = \begin{Bmatrix} \Lambda_\gamma \\ \Lambda_\kappa \end{Bmatrix} = \begin{Bmatrix} \Lambda_\vartheta + \frac{1}{2}(\Lambda_\varphi u) \otimes \Lambda_\varphi \\ \Lambda_\kappa \end{Bmatrix} \tag{10}$$

in which

$$\Lambda_\gamma \mathbf{u} = \{\gamma_{\alpha\beta}(u_\alpha, w)\}$$
$$\Lambda_\kappa \mathbf{u} = \{\kappa_{\alpha\beta}(u_\alpha, w)\}$$
$$\Lambda_\vartheta \mathbf{u} = \{\vartheta_{\alpha\beta}(u_\alpha, w)\}$$
$$\Lambda_\varphi \mathbf{u} = \{\varphi_\alpha(u_\alpha, w)\}.$$

Then the strain-displacement relations (4) can be written in the following form

$$\Lambda(\mathbf{u})\mathbf{u} - \mathbf{E}(\mathbf{u}) = 0 \quad \text{in } S. \tag{11}$$

According to Gao and Strang (1988), the nonlinear operator Λ_γ can be decomposed into two parts:

$$\Lambda(\mathbf{u}) = \Lambda_T(\mathbf{u}) + \Lambda_N(\mathbf{u}) = \begin{Bmatrix} \Lambda_t(\mathbf{u}) \\ \Lambda_\kappa \end{Bmatrix} + \begin{Bmatrix} \Lambda_n(\mathbf{u}) \\ 0 \end{Bmatrix} \tag{12}$$

where

$$\Lambda_t(\mathbf{u}) = \Lambda_\vartheta + (\Lambda_\varphi \mathbf{u}) \otimes \Lambda_\varphi \tag{13}$$

$$\Lambda_n(\mathbf{u}) = -\tfrac{1}{2}(\Lambda_\varphi \mathbf{u}) \otimes \Lambda_\varphi. \tag{14}$$

We can see here Λ_n is a symmetric and quadratic operator:

$$\Lambda_n(\mathbf{u})\mathbf{u} = \{-\tfrac{1}{2}\varphi_\alpha(\mathbf{u})\varphi_\beta(\mathbf{u})\}.$$

It will play an important part in the formulation of the complementary energy.

According to the complementarity of energy, the conjugate space $\mathscr{U}^*$ of the admissible displacement space $\mathscr{U}$ should be the space of admissible forces. We put $\mathscr{T} = \mathscr{U}^*$. The element $\mathbf{t} \in \mathscr{T}$ denote the distributed loads: $\mathbf{t} = \{\mathbf{p} \text{ in } S;\ \mathbf{P},\ M_n \text{ on } \Gamma\}$. Similarly, the conjugate space $\mathscr{S} = \mathscr{E}^*$ of $\mathscr{E}$ should be the admissible stress space. The element $\mathbf{T} \in \mathscr{S}$ denotes the generalized stress:

$$\mathbf{T} = \begin{Bmatrix} \mathbf{N} \\ \mathbf{M} \end{Bmatrix}.$$

The billinear form $\langle \mathbf{E}, \mathbf{T} \rangle = \gamma_{\alpha\beta} N^{\alpha\beta} + \kappa_{\alpha\beta} M^{\alpha\beta}$ puts $\mathscr{E}$ and $\mathscr{S}$ in duality, and we write

$$\langle \mathbf{E}, \mathbf{T} \rangle_S = \iint_S \langle \mathbf{E}, \mathbf{T} \rangle \, \mathrm{d}S. \tag{15}$$

In the same way, the bilinear form $(\mathbf{u}, \mathbf{t}): \mathscr{U} \times \mathscr{T} \to \mathscr{R}$ puts $\mathscr{U}$ and $\mathscr{T}$ in duality, and we define

$$(\mathbf{u}, \mathbf{t})_S = \iint_S (p^\alpha u_\alpha + pw) \, \mathrm{d}S + \int_\Gamma \left(P^\alpha u_\alpha + Pw - M_n \frac{\partial w}{\partial n} \right) \mathrm{d}\Gamma. \tag{16}$$

Using the Gauss-Green law, it is easy to prove that

$$\langle \Lambda(\mathbf{u})\mathbf{u}, \mathbf{T}\rangle_S = \langle \Lambda_t(\mathbf{u})\mathbf{u}, \mathbf{N}\rangle_S + \langle \Lambda_n(\mathbf{u})\mathbf{u}, \mathbf{N}\rangle_S + \langle \Lambda_\kappa \mathbf{u}, \mathbf{M}\rangle_S$$
$$= \langle \mathbf{u}, \Lambda_t^*(\mathbf{u})\mathbf{N}\rangle_S + \langle \mathbf{u}, \Lambda_\kappa^* \mathbf{M}\rangle_S - G(\mathbf{u}, \mathbf{N}) \tag{17}$$

in which Λ_t^* and Λ_κ^* are the conjugate operators of Λ_t and Λ_κ, respectively:

$$\Lambda_t^*(\mathbf{u})\mathbf{N} = \begin{cases} \{-N^{\alpha\beta}_{|\beta} + b^\alpha_\lambda N^{\lambda\beta}\varphi_\beta(\mathbf{u})\} & \text{in } S \\ -b_{\alpha\beta}N^{\alpha\beta} - (N^{\alpha\beta}\varphi_\beta(\mathbf{u}))_{|\alpha} & \text{in } S \\ \{N^{\alpha\beta}n_\beta\} & \text{on } \Gamma_t \\ N^{\lambda\beta}\varphi_\beta n_\alpha & \text{on } \Gamma_t \end{cases} \tag{18}$$

$$\Lambda_\kappa^* \mathbf{M} = \begin{cases} \{[b^\alpha_\beta(M^{\lambda\beta} + M^{\beta\lambda})]_{|\lambda} - b^\alpha_{\lambda|\beta}M^{\lambda\beta}\} & \text{in } S \\ -M^{\alpha\beta}_{|\alpha\beta} + b^\lambda_\alpha b_{\lambda\beta}M^{\alpha\beta} & \text{in } S \\ \{-b^\alpha_\beta(M^{\lambda\beta} + M^{\beta\lambda})n_\lambda\} & \text{on } \Gamma_t \\ M^{\alpha\beta}_{|\alpha\beta}n_\alpha + \dfrac{\partial}{\partial s}(M^{\alpha\beta}n_\alpha t_\beta) & \text{on } \Gamma_t \\ M^{\alpha\beta}n_\alpha n_\beta & \text{on } \Gamma_t \end{cases} \tag{19}$$

$G(\mathbf{u}, \mathbf{N})$ is the so-called dual gap function (Gao, 1989; Gao and Strang, 1988):

$$G(\mathbf{u}, \mathbf{N}) = \langle -\Lambda_n(\mathbf{u})\mathbf{u}, \mathbf{N}\rangle_S = \iint_S \tfrac{1}{2}\varphi_\alpha(\mathbf{u})\varphi_\beta(\mathbf{u})N^{\alpha\beta}\,\mathrm{d}S. \tag{20}$$

Let

$$\bar{\mathbf{t}} = \begin{cases} \bar{p}^\alpha & \text{in } S \\ \bar{p} & \text{in } S \\ \bar{P}^\alpha & \text{on } \Gamma_t \\ \bar{P} & \text{on } \Gamma_t \\ \bar{M}_n & \text{on } \Gamma_t \end{cases}$$

$$\Lambda_T^*(\mathbf{u}) = \begin{Bmatrix} \Lambda_t^*(\mathbf{u}) \\ \Lambda_\kappa^* \end{Bmatrix}.$$

The equilibrium equations (7) and the statical boundary conditions (9) can be written in the following abstract form:

$$\Lambda_t^*(\mathbf{u})\mathbf{N} + \Lambda_\kappa^*\mathbf{M} - \bar{\mathbf{t}} = 0 \quad \text{on } S \cup \Gamma_t, \tag{21a}$$

or

$$\Lambda_T^*(\mathbf{u})\mathbf{T} - \bar{\mathbf{t}} = 0 \quad \text{on } S \cup \Gamma_t. \tag{21b}$$

Let $\mathscr{U}_a \subset \mathscr{U}$ be a kinematically admissible space:

$$\mathscr{U}_a = \left\{\mathbf{u} \in \mathscr{U} \mid u_\alpha = w = \frac{\partial w}{\partial n} = 0 \quad \text{on } \Gamma_u\right\}. \tag{22}$$

The boundary value problem for geometrical nonlinear thin elastic shells is finding the displacements $\mathbf{u} \in \mathscr{U}_a$, such that

$$\begin{cases} \Lambda(\mathbf{u})\mathbf{u}-\mathbf{E}=0 & \text{in } S \\ \Lambda_T^*(\mathbf{u})\mathbf{T}-\bar{\mathbf{t}}=0 & \text{in } S\cup\Gamma_t \\ \mathbf{T}-\dfrac{\partial W}{\partial \mathbf{E}}=0 & \text{in } S \end{cases}. \tag{23}$$

The external potential done by the distributed load $\bar{\mathbf{t}}$ is a linear functional of $\mathbf{u}$ over $\mathscr{U}_a$:

$$F(\mathbf{u})=(\mathbf{u},-\bar{\mathbf{t}})_S=-\iint_S(\bar{p}^{\alpha}u_{\alpha}+\bar{p}w)\,\mathrm{d}S \\ -\int_{\Gamma_t}\left(\bar{P}^{\alpha}u_{\alpha}+\bar{P}w-\bar{M}_n\frac{\partial w}{\partial n}\right)\mathrm{d}\Gamma. \tag{24}$$

The total potential energy functional $\Pi:\mathscr{E}\times\mathscr{U}_a\rightarrow\mathbb{R}$ should be:

$$\Pi(\mathbf{E},\mathbf{u})=\iint_S W(\gamma,\kappa)\,\mathrm{d}S+F(\mathbf{u}). \tag{25}$$

Substituting the geometrical relation (11) into (25), we write

$$\Xi(\mathbf{u})=\Pi(\Lambda\mathbf{u},\mathbf{u}). \tag{26}$$

Then we have the primal problem for geometrically nonlinear thin elastic shells: to find $\bar{\mathbf{u}}\in\mathscr{U}_a$ such that

$$\Xi(\bar{\mathbf{u}})=\operatorname*{sta}_{\mathbf{u}\in\mathscr{U}_a}\Xi(\mathbf{u}). \tag{27}$$

It is easy to prove that for any given $\mathbf{u}\in\mathscr{U}_a$, the stationary points $\bar{\mathbf{u}}$ of Ξ satisfy

$$\delta\Xi(\bar{\mathbf{u}})=0\Leftrightarrow\Lambda_T^*(\bar{\mathbf{u}})\frac{\partial W(\Lambda\bar{\mathbf{u}})}{\partial(\Lambda\bar{\mathbf{u}})}-\bar{\mathbf{t}}=0 \tag{28}$$

i.e. the stationary points $\bar{\mathbf{u}}$ solve the boundary value problem (23).

We should emphasize that although the stored energy function $W(\mathbf{E}):\mathscr{E}\rightarrow\mathbb{R}$ is convex with respect to the strain tensor $\mathbf{E}$, $W(\Lambda\mathbf{u}):\mathscr{U}\rightarrow\mathbb{R}$ is not sure convex with respect to the displacement $\mathbf{u}$ due to the nonlinearity of geometrical operator Λ. It has been proved Gao (1989) that if the gap function G satisfies

$$G(\mathbf{u},\bar{\mathbf{N}}(\bar{\mathbf{u}}))=\iint_S\frac{1}{2}\frac{\partial W(\Lambda\bar{\mathbf{u}})}{\partial(\Lambda_\gamma\bar{\mathbf{u}})}:(\Lambda_\varphi\mathbf{u}\otimes\Lambda_\varphi\mathbf{u})\,\mathrm{d}S\geqslant 0\quad\forall\mathbf{u}\in\mathscr{U}_a \tag{29}$$

the total potential energy $\Xi:\mathscr{U}_a\rightarrow\mathbb{R}$ is convex. In this case, we have the minimal potential principle:

For any given $\mathbf{u}\in\mathscr{U}_a$, if the gap function G satisfies inequality (29), then the stationary points $\bar{\mathbf{u}}$ of Ξ minimize Ξ:

$$\Xi(\bar{\mathbf{u}}) = \inf_{\mathbf{u}\in\mathscr{U}_a} \Xi(\mathbf{u}). \tag{30}$$

4. CONJUGATE TRANSFORMATION AND COMPLEMENTARY ENERGY

According to the theory of convex analysis, the conjugate function of the stored energy W can be given by using Legendre-Fenchel transformation:

$$W^*(\mathbf{T}) = \sup_{\mathbf{E}\in\mathscr{E}} \{\langle \mathbf{E}, \mathbf{T}\rangle - W(\mathbf{E})\}$$
$$= \tfrac{1}{2}H^{-1}_{\alpha\beta\lambda\mu}N^{\alpha\beta}N^{\lambda\mu} + \tfrac{1}{2}h^{-1}_{\alpha\beta\lambda\mu}M^{\alpha\beta}M^{\lambda\mu}. \tag{31}$$

Obviously $W^*: \mathscr{S} \to \mathbb{R}$ is convex and quadratic. The conjugate function of $F(\mathbf{u})$ is given by:

$$F^*(-\Lambda_T^*(\bar{\mathbf{u}})\mathbf{T}) = \sup_{\mathbf{u}\in\mathscr{U}_a} \{(\mathbf{u}, -\Lambda_T^*(\bar{\mathbf{u}})\mathbf{T})_S - F(\mathbf{u})\}$$
$$= \sup_{\mathbf{u}\in\mathscr{U}_a} \{(\mathbf{u}, -\Lambda_T^*(\mathbf{u})\mathbf{T} + \bar{\mathbf{t}})_S\}$$
$$= \begin{cases} 0 & \text{if } \Lambda_T^*(\mathbf{u})\mathbf{T} - \bar{\mathbf{t}} = 0 \quad \text{on } S \cup \Gamma_t \\ +\infty & \text{otherwise.} \end{cases} \tag{32}$$

So the conjugate functional of the total potential energy Π should be

$$\Pi^*(\mathbf{T}, -\Lambda_T^*(\bar{\mathbf{u}})\mathbf{T}) = \iint_S W^*(\mathbf{T})\,\mathrm{d}S + F^*(-\Lambda_T^*(\bar{\mathbf{u}})\mathbf{T}). \tag{33}$$

However, unfortunately for geometrical nonlinear problems, the conjugate functional Π^* of the total potential energy is not the total complementary energy functional (Gao and Strang, 1988). The difference is just the gap function G. Hence, the total complementary energy functional for geometrical nonlinear thin elastic shells is:

$$\Pi_c^*(\mathbf{T}, -\Lambda^*(\mathbf{u})\mathbf{T}) = \Pi^*(\mathbf{T}, -\Lambda_T^*(\mathbf{u})\mathbf{T}) + G(\mathbf{u}, \mathbf{N})$$
$$= \iint_S W^*(\mathbf{T})\,\mathrm{d}S + \iint_S \tfrac{1}{2}\varphi_\alpha(\mathbf{u})\varphi_\beta(\mathbf{u})N^{\alpha\beta}\,\mathrm{d}S + F^*(-\Lambda_T^*(\mathbf{u})\mathbf{T}). \tag{34}$$

We note that the variational arguments in Π_c^* is not only the stress tensor

$$\mathbf{T} = \begin{Bmatrix} \mathbf{N} \\ \mathbf{M} \end{Bmatrix},$$

but also the displacement

$$\mathbf{u} = \begin{Bmatrix} u_\alpha \\ w \end{Bmatrix}$$

due to the nonlinearity of the geometric operator Λ.

Let $\mathscr{S}_a \subset \mathscr{U} \times \mathscr{S}$ be a statically admissible space:

$$\mathscr{S}_a = \{(\mathbf{u}, \mathbf{T}) \in \mathscr{U} \times \mathscr{S} \mid \Lambda_T^*(\mathbf{u})\mathbf{T} - \bar{\mathbf{t}} = 0 \quad \text{in } S \cup \Gamma_t\} \tag{35}$$

then we have the stationary complementary energy principle of the geometrical nonlinear elastic shells:

Among all the $(\mathbf{u}, \mathbf{T}) \in \mathcal{S}_a$, *the stationary points* $(\bar{\mathbf{u}}, \bar{\mathbf{T}})$ *of* Π_c^* *solve the boundary value problem (23), i.e.*

$$\delta\Pi_c^*(\bar{\mathbf{T}}, -\Lambda^*(\bar{\mathbf{u}})\bar{\mathbf{T}}) = 0 \Leftrightarrow \begin{cases} \Lambda\bar{\mathbf{u}} - \dfrac{\partial W^*(\bar{\mathbf{T}})}{\partial \bar{\mathbf{T}}} = 0 & \text{in } S \\ \bar{u}_\alpha = \bar{w} = \dfrac{\partial \bar{w}}{\partial n} = 0 & \text{on } \Gamma_u \end{cases}. \tag{36}$$

For a given stationary point $\bar{\mathbf{u}}$ of Π_c^*, over $\mathcal{S}_a$, let

$$\begin{aligned}\Xi^*(\mathbf{T}) &= -\Pi_c^*(\mathbf{T}, -\Lambda^*(\bar{\mathbf{u}})\mathbf{T}) \\ &= -\iint_S W^*(\mathbf{T})\,\mathrm{d}S - \iint_S \tfrac{1}{2}\varphi_\alpha(\bar{\mathbf{u}})\varphi_\beta(\bar{\mathbf{u}})N^{\alpha\beta}\,\mathrm{d}S.\end{aligned} \tag{37}$$

Then the dual variational principle for nonlinear shell analysis can be described as follows:

Theorem 1. *Among all statically admissible fields* $\mathbf{T} \in \mathcal{S}_a$, *the solutions* $\bar{\mathbf{T}}$ *of the boundary value problem (23) maximize* $\Xi^*(\mathbf{T})$, *i.e.*

$$\Xi^*(\bar{\mathbf{T}}) = \sup_{\mathbf{T}\in\mathcal{S}_a} \Xi^*(\mathbf{T}). \tag{38}$$

Proof. Considering the convexity of the complementary energy W^*, the constitutive equation

$$\bar{\mathbf{E}} = \frac{\partial W^*(\bar{\mathbf{T}})}{\partial \bar{\mathbf{T}}}$$

yields the following inequality:

$$W^*(\mathbf{T}) - W^*(\bar{\mathbf{T}}) \geqslant \langle \bar{\mathbf{E}}, \mathbf{T} - \bar{\mathbf{T}} \rangle \quad \forall\, \mathbf{T} \in \mathcal{S} \tag{39a}$$

i.e.

$$W^*(\mathbf{N}, \mathbf{M}) - W^*(\bar{\mathbf{N}}, \bar{\mathbf{M}}) \geqslant \langle \bar{\gamma}, \mathbf{N} - \bar{\mathbf{N}} \rangle + \langle \bar{\kappa}, \mathbf{M} - \bar{\mathbf{M}} \rangle \quad \forall\, (\mathbf{N}, \mathbf{M}) \in \mathcal{S}. \tag{39b}$$

Substituting the geometric relation $\bar{\mathbf{E}} = \Lambda\bar{\mathbf{u}}$ into (39) gives

$$\begin{aligned}W^*(\mathbf{N}, \mathbf{M}) - W^*(\bar{\mathbf{N}}, \bar{\mathbf{M}}) \geqslant{}& \langle \Lambda_t(\bar{\mathbf{u}})\bar{\mathbf{u}}, \mathbf{N} - \bar{\mathbf{N}} \rangle + \langle \Lambda_\kappa\bar{\mathbf{u}}, \mathbf{M} - \bar{\mathbf{M}} \rangle \\ &+ \langle \Lambda_n(\bar{\mathbf{u}})\bar{\mathbf{u}}, \mathbf{N} - \bar{\mathbf{N}} \rangle \quad \forall\, (\mathbf{N}, \mathbf{M}) \in \mathcal{S}.\end{aligned} \tag{40}$$

Integrating (40) over the midsurface S and using Gauss–Green transformation, we obtain

$$\begin{aligned}\iint_S W^*(\mathbf{N}, \mathbf{M})\,\mathrm{d}S - \iint_S W^*(\bar{\mathbf{N}}, \bar{\mathbf{M}})\,\mathrm{d}S \geqslant{}& (\bar{\mathbf{u}}, \Lambda_t^*(\bar{\mathbf{u}})\mathbf{N} + \Lambda_\kappa^*(\bar{\mathbf{u}})\mathbf{M})_S \\ &+ \langle \Lambda_n(\bar{\mathbf{u}})\bar{\mathbf{u}}, \mathbf{N} \rangle_S - (\bar{\mathbf{u}}, \Lambda_t^*(\bar{\mathbf{u}})\bar{\mathbf{N}} + \Lambda_\kappa^*(\bar{\mathbf{u}})\bar{\mathbf{M}})_S - \langle \Lambda_n(\bar{\mathbf{u}})\bar{\mathbf{u}}, \bar{\mathbf{N}} \rangle \\ &= (\bar{\mathbf{u}}, \Lambda_\dagger^*(\bar{\mathbf{u}})\mathbf{T} - \bar{\mathbf{t}})_S - G(\bar{\mathbf{u}}, \mathbf{N}) + G(\bar{\mathbf{u}}, \bar{\mathbf{N}}) \quad \forall\, \mathbf{T} \in \mathcal{S}.\end{aligned} \tag{41}$$

So for any given statically admissible stresses $\mathbf{T} \in \mathcal{S}_a$, we have

$$\Xi^*(\mathbf{T}) - \Xi^*(\bar{\mathbf{T}}) \leqslant 0 \quad \forall\, \mathbf{T} \in \mathcal{S}_a \tag{42}$$

which means that $\bar{\mathbf{T}}$ maximizes Ξ^* over $\mathcal{S}_a$.

Furthermore, we can prove the following complementary extremum principle (two-fields complementary variational extremum principle):

Theorem 2. *If the gap function* $G(\mathbf{u}, \mathbf{N}) \geqslant 0$ *for any given* $(\mathbf{u}, \mathbf{N}) \in \mathscr{S}_a$, *the solutions* $(\bar{\mathbf{u}}, \bar{\mathbf{T}})$ *of the boundary value problem (23) minimize* Π_c^*, *i.e.*

$$\Pi_c^*(\bar{\mathbf{T}}, -\Lambda^*(\bar{\mathbf{u}})\bar{\mathbf{T}}) = \inf_{(\mathbf{u},\mathbf{T})\in\mathscr{S}_a} \Pi_c^*(\mathbf{T}, -\Lambda^*(\mathbf{u})\mathbf{T}). \tag{43}$$

Proof. With $\delta\mathbf{u} = \bar{\mathbf{u}} - \mathbf{u}$,

$$\begin{aligned}\mathbf{E}(\bar{\mathbf{u}}) &= \mathbf{E}(\mathbf{u}) + \Lambda_T(\mathbf{u})\delta\mathbf{u} - \Lambda_N(\delta\mathbf{u})\delta\mathbf{u}\\ &= \Lambda_T(\mathbf{u})\mathbf{u} + \Lambda_N(\mathbf{u})\mathbf{u} + \Lambda_T(\mathbf{u})\delta\mathbf{u} - \Lambda_N(\delta\mathbf{u})\delta\mathbf{u}\\ &= \Lambda_T(\mathbf{u})\bar{\mathbf{u}} + \Lambda_N(\mathbf{u})\mathbf{u} - \Lambda_N(\delta\mathbf{u})\delta\mathbf{u}.\end{aligned}$$

It means

$$\begin{aligned}\gamma_{\alpha\beta}(\bar{u}_\alpha, \bar{w}) &= \vartheta_{\alpha\beta}(\bar{u}_\alpha, \bar{w}) + \varphi_\alpha(u_\alpha, w)\varphi_\beta(\bar{u}_\alpha, \bar{w}) - \tfrac{1}{2}\varphi_\alpha(u_\alpha, w)\varphi_\beta(u_\alpha, w)\\ &\quad + \tfrac{1}{2}\varphi_\alpha(\delta u_\alpha, \delta w)\varphi_\beta(\delta u_\alpha, \delta w)\\ \kappa_{\alpha\beta}(\bar{u}_\alpha, \bar{w}) &= \kappa_{\alpha\beta}(\bar{u}_\alpha, \bar{w}).\end{aligned}$$

Substituting into eqn (39), we obtain

$$\begin{aligned}W^*(\mathbf{T}) - W^*(\bar{\mathbf{T}}) \geqslant \langle\Lambda_T(\mathbf{u})\bar{\mathbf{u}}, \mathbf{T}\rangle + \langle\Lambda_N(\mathbf{u})\mathbf{u}, \mathbf{T}\rangle - \langle\Lambda_N(\delta\mathbf{u})\delta\mathbf{u}, \mathbf{T}\rangle&\\ - \langle\Lambda_T(\bar{\mathbf{u}})\bar{\mathbf{u}}, \bar{\mathbf{T}}\rangle - \langle\Lambda_N(\bar{\mathbf{u}})\bar{\mathbf{u}}, \bar{\mathbf{T}}\rangle \quad \forall\,\mathbf{T}\in\mathscr{S}.&\end{aligned} \tag{44}$$

Integrating (44) and using Gauss–Green transformation, we have

$$\begin{aligned}\iint_S W^*(\mathbf{T})\,\mathrm{d}S + \langle -\Lambda_n(\mathbf{u})\mathbf{u}, \mathbf{N}\rangle_S - \iint_S W^*(\bar{\mathbf{T}})\,\mathrm{d}S - \langle -\Lambda_n(\bar{\mathbf{u}})\bar{\mathbf{u}}, \bar{\mathbf{N}}\rangle_S&\\ \geqslant (\bar{\mathbf{u}}, \Lambda_T^*(\mathbf{u})\mathbf{T})_S - (\bar{\mathbf{u}}, \Lambda_T^*(\bar{\mathbf{u}})\bar{\mathbf{T}})_S - \langle\Lambda_n(\delta\mathbf{u})\delta\mathbf{u}, \mathbf{N}\rangle_S.&\end{aligned} \tag{45}$$

For any given $(\mathbf{u}, \mathbf{T}) \in \mathscr{S}_a$, inequality (45) yields

$$\Pi_c^*(\mathbf{T}, -\Lambda^*(\mathbf{u})\mathbf{T}) - \Pi_c^*(\bar{\mathbf{T}}, -\Lambda^*(\bar{\mathbf{u}})\bar{\mathbf{T}}) \geqslant G(\delta\mathbf{u}, \mathbf{N}) \quad \forall\,(\mathbf{u}, \mathbf{T}) \in \mathscr{S}_a. \tag{46}$$

By assumption $G(\mathbf{u}, \mathbf{N}) \geqslant 0$ for any $(\mathbf{u}, \mathbf{T}) \in \mathscr{U} \times \mathscr{S}$, we have

$$\Pi_c^*(\mathbf{T}, -\Lambda^*(\mathbf{u})\mathbf{T}) - \Pi_c^*(\bar{\mathbf{T}}, -\Lambda^*(\bar{\mathbf{u}})\bar{\mathbf{T}}) \geqslant 0 \quad \forall\,(\mathbf{u}, \mathbf{T}) \in \mathscr{S}_a$$

which means that $(\bar{\mathbf{u}}, \bar{\mathbf{T}})$ minimize Π_c^* over $\mathscr{S}_a$.

5. GENERALIZED SADDLE POINT VARIATIONAL PRINCIPLE

The equilibrium constraint in extremum principle (43) can be relaxed by Lagrangian $L: \mathscr{U} \times \mathscr{S} \to \mathbb{R}$:

$$\begin{aligned}L(\mathbf{u}, \mathbf{T}) &= (\mathbf{u}, \Lambda_T^*(\mathbf{u})\mathbf{T} - \bar{\mathbf{t}})_S - \iint_S W^*(\mathbf{T})\,\mathrm{d}S - G(\mathbf{u}, \mathbf{N})\\ &= \langle\Lambda_T(\mathbf{u})\mathbf{u}, \mathbf{T}\rangle_S - \iint_S W^*(\mathbf{T})\,\mathrm{d}S + \langle\Lambda_N(\mathbf{u})\mathbf{u}, \mathbf{T}\rangle_S - (\mathbf{u}, \bar{\mathbf{t}})_S\\ &= \langle\Lambda(\mathbf{u})\mathbf{u}, \mathbf{T}\rangle_S - \iint_S W^*(\mathbf{T})\,\mathrm{d}S - (\mathbf{u}, \bar{\mathbf{t}})_S.\end{aligned} \tag{47}$$

The generalized variational principle for geometrical nonlinear elastic thin shells states that:

Among all $\mathbf{u} \in \mathscr{U}_a$ *and* $\mathbf{T} \in \mathscr{S}$, *the stationary points* $(\bar{\mathbf{u}}, \bar{\mathbf{T}})$ *of* $L(\mathbf{u}, \mathbf{T})$ *solve the boundary value problem (23), i.e.*

$$\delta L(\bar{\mathbf{u}}, \bar{\mathbf{T}}) = 0 \Leftrightarrow \begin{cases} \Lambda(\bar{\mathbf{u}})\bar{\mathbf{u}} - \dfrac{\partial W^*(\bar{\mathbf{T}})}{\partial \bar{\mathbf{T}}} = 0 & \text{in } S \\ \Lambda_T^*(\bar{\mathbf{u}})\bar{\mathbf{T}} - \bar{\mathbf{t}} = 0 & \text{in } S \cup \Gamma_t \end{cases}. \tag{48}$$

The following theorem states the saddle point property of Lagrangian $L(\mathbf{u}, \mathbf{T})$.

Theorem 3. *Among all* $\mathbf{u} \in \mathscr{U}_a$ *and* $\mathbf{T} \in \mathscr{S}$, *if the gap function G satisfies* $G(\mathbf{u}, \mathbf{T}) \geq 0$, *then the solutions* $(\bar{\mathbf{u}}, \bar{\mathbf{T}})$ *of the boundary value problem (23) are the saddle points of* $L(\mathbf{u}, \mathbf{T})$, *i.e.*

$$L(\bar{\mathbf{u}}, \bar{\mathbf{T}}) = \inf_{\mathbf{u} \in \mathscr{U}_a} \sup_{\mathbf{T} \in \mathscr{S}} L(\mathbf{u}, \mathbf{T}). \tag{49}$$

Proof. Since

$$\sup_{\mathbf{T} \in \mathscr{S}} L(\mathbf{u}, \mathbf{T}) = \sup_{\mathbf{T} \in \mathscr{S}} \left\{ \iint_S [\langle \mathbf{E}(\mathbf{u}), \mathbf{T} \rangle - W^*(\mathbf{T})] \, \mathrm{d}S \right\} + (\mathbf{u}, -\bar{\mathbf{t}})_S$$

$$= \iint_S W(\mathbf{E}(\mathbf{u})) \, \mathrm{d}S + (\mathbf{u}, -\bar{\mathbf{t}})_S = \Xi(\mathbf{u}). \tag{50}$$

Recalling the extremum potential principle (30), we have

$$\inf_{\mathbf{u} \in \mathscr{U}_a} \sup_{\mathbf{T} \in \mathscr{S}} L(\mathbf{u}, \mathbf{T}) = \inf_{\mathbf{u} \in \mathscr{U}_a} L(\mathbf{u}, \bar{\mathbf{T}}) = \inf_{\mathbf{u} \in \mathscr{U}_a} \Xi(\mathbf{u}) = L(\bar{\mathbf{u}}, \bar{\mathbf{T}}).$$

6. CONCLUDING REMARK

The results presented above show that in the theory of geometrical nonlinear elastic shells, there exists a complementary gap function between the total potential energy functional $\Xi(\mathbf{u})$ and the total complementary energy functional $\Xi^*(\mathbf{T})$. This gap function provides a global extremal criteria for the primal-dual variational problems, i.e. for any given admissible $\mathbf{u} \in \mathscr{U}$ and $\mathbf{N} \in \mathscr{S}$, if

$$G(\mathbf{u}, \mathbf{N}) = \iint_S \tfrac{1}{2}\varphi_\alpha(\mathbf{u})\varphi_\beta(\mathbf{u})N^{\alpha\beta} \, \mathrm{d}S \geq 0$$

then the $\Xi(\mathbf{u})$ and $\Xi^*(\mathbf{T})$ are convex and concave functionals, respectively. In this case, the boundary value problem (23) is stable. Otherwise, the shell may be instable. Theorem 3 suggests a generalized saddle point variational principle for problem (23). By using finite element method, this principle gives rise to a min–max nonlinear optimization problem. Since the equilibrium constraint for $\mathbf{u}$ and $\mathbf{T}$ is relaxed, it is easy to choose try functions in engineering problems.

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REFERENCES

Gao, Y. (1990). On the extremum potential variational principle for geometrical nonlinear thin elastic shells, to appear in *Scientia Sinica* (*A*).

Gao, Y. (1989). Opposite principles for nonlinear conservative systems. *Adv. in Appl. Math.* **10**(3).

Gao, Y. and Strang, G. (1990). Dual extremum principles in finite deformation elastoplastic analysis, to appear in *Acta Appl. Math.*

Gao, Y. and Strang, G. (1989). Geometrical nonlinearity: Potential energy, complementary energy and the gap function, *17th IUTAM, Grenoble, France. Quart. Appl. Math.* **47**(3), 487–504.

Iura, M. (1986). A generalized variational principle for thin elastic shells with finite rotations. *Int. J. Solids Struct.* **22**(2), 141–154.

Koiter, W. T. (1986). On the nonlinear theory of thin elastic shells. *Proc. K. Ned. Akad. Wet., Ser.* **B69**, 1–54–54.

Schmidt, R. and Pietraszkiewicz, W. (1981). Variational principles in the geometrically non-linear theory of shells undergoing moderate rotations. *Ing. Arch.* **50**(3), 181.

Stumpf, H. (1979). The derivation of the dual extremum and complementary stationary principles in geometrical nonlinear shell theory. *Ing. Arch.* **48**, 221–237.

Szwabowicz, M. L. (1986). Variational formulation in the geometrically nonlinear thin elastic shell theory. *Int. J. Solids Struct.* **11**(22), 1161–1175.

Wempner, G. (1986). A general theory of shells and the complementary potentials. *J. Appl. Mech.* **53**, 881–885.